APPENDIX A

REALIZABILITY OF FILTERS

Given a certain amplitude characteristic, the question arises as to whether or not it can be realized by means of a physical filter. For example, is there any physical network whose absolute value is a gaussian-error curve? The only restriction on a "physical network" is that it show no response to an applied signal until the input switch is closed; i.e., that the Fourier transform be zero for time $t < 0$.

A complete and very simple answer to the question of realizability is provided by a theorem of Paley and Wiener, drawn from the theory of Fourier transforms in the complex domain.

The result for the gaussian-error curve is that it is not realizable.

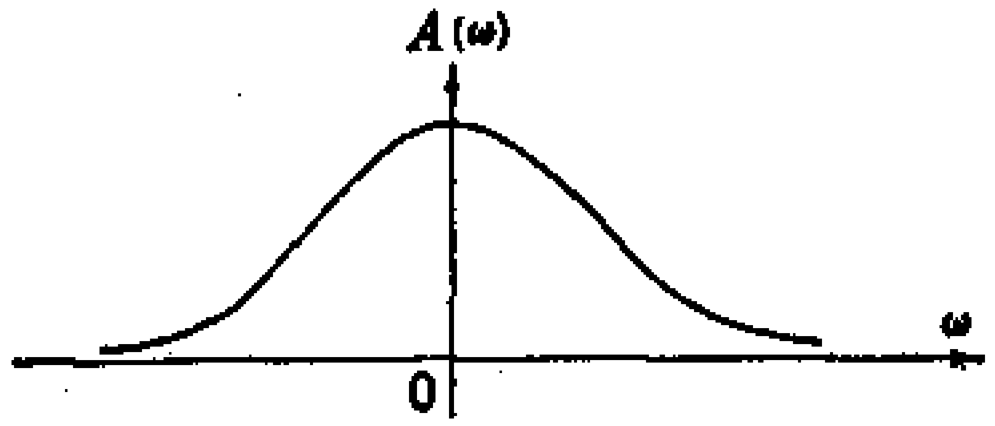


Fig. A·1.—Gaussian-error-curve filter, $A(\omega) = e^{-\omega^2}$.

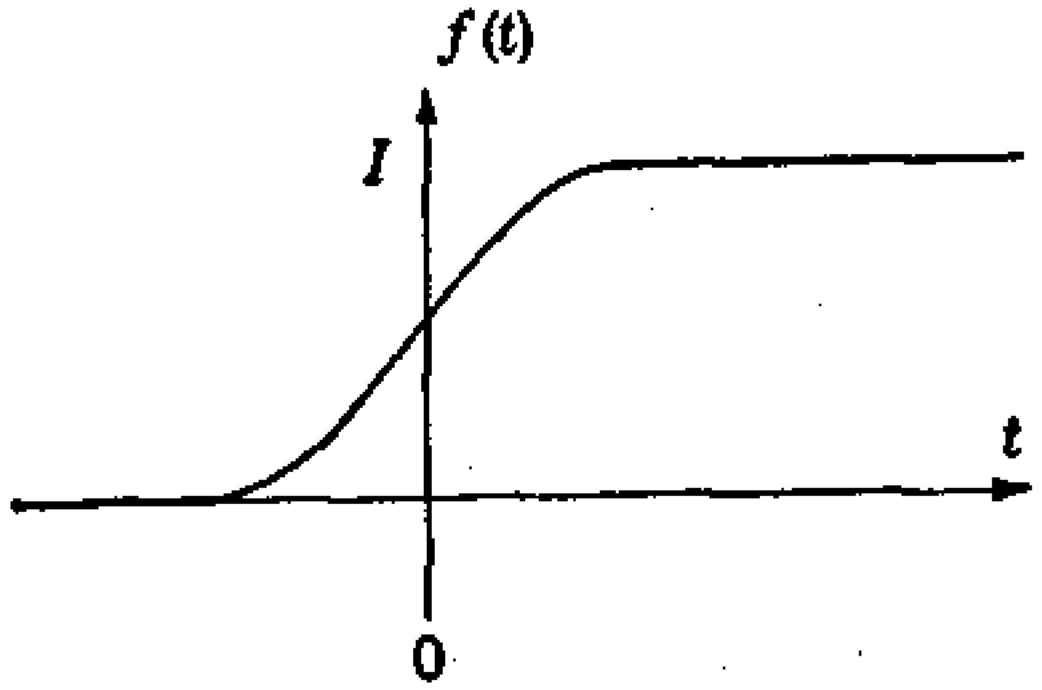


Fig. A·2.—Step-function response of Gaussian-error-curve filter.

A·1. The Paley-Wiener Criterion.—If it were possible for a filter to have a gaussian-error curve $A(\omega) = e^{-\omega^2}$ as amplitude characteristic (Fig. A·1), and zero phase lag, so that the complex system function is $H(\omega) = e^{-\omega^2}$, then the step-function response

$$f(t) = \frac{1}{2\pi} \int_{-\infty - jc}^{\infty - jc} \frac{H(\omega)}{j\omega} e^{j\omega t}\, d\omega$$

of such a filter is the error function (Fig. A·2)

$$f(t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{t} e^{-t^2/4}\, dt.$$

Now consider an idealized low-pass filter, having for amplitude characteristic the function $A(\omega) = 1$ for $-1 < \omega < 1$ and $A(\omega) = 0$ otherwise (Fig. A·3), and zero phase lag, so that the complex system function is $H(\omega) = 1$ for $-1 < \omega < 1$, and $H(\omega) = 0$ otherwise. Then the step-function response is the sine-integral function (Fig. A·4)

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{t} \frac{\sin t}{t}\, dt.$$

There is something very disquieting in Figs. A·2 and A·4, in that they show response for time $t < 0$ although a physical system obviously cannot react to a step function before the step function has even been applied.

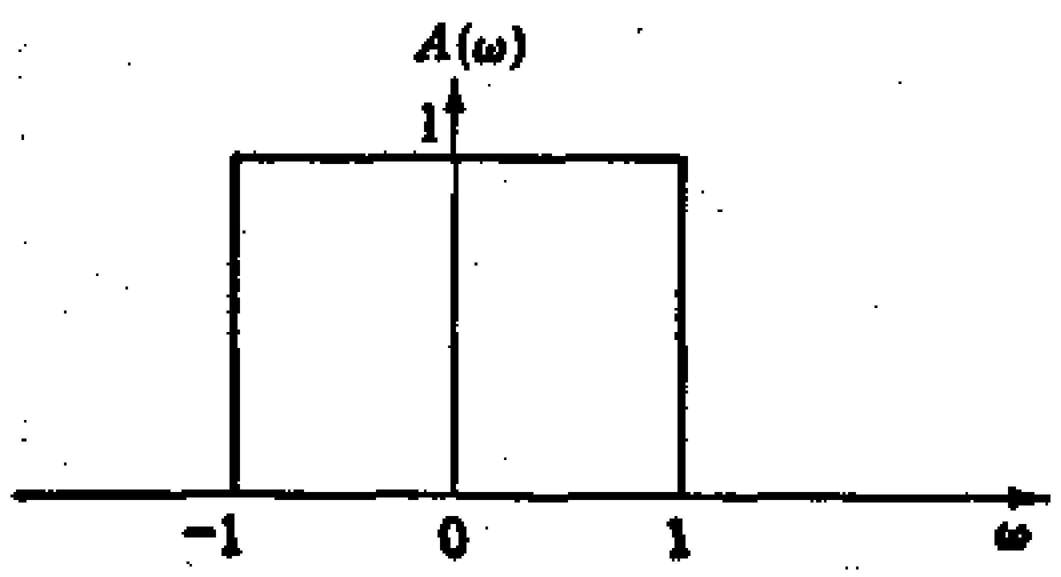


FIG. A·3.—Idealized low-pass filter $A(\omega) = 1$ for $-1 < \omega < 1$, $= 0$ otherwise.

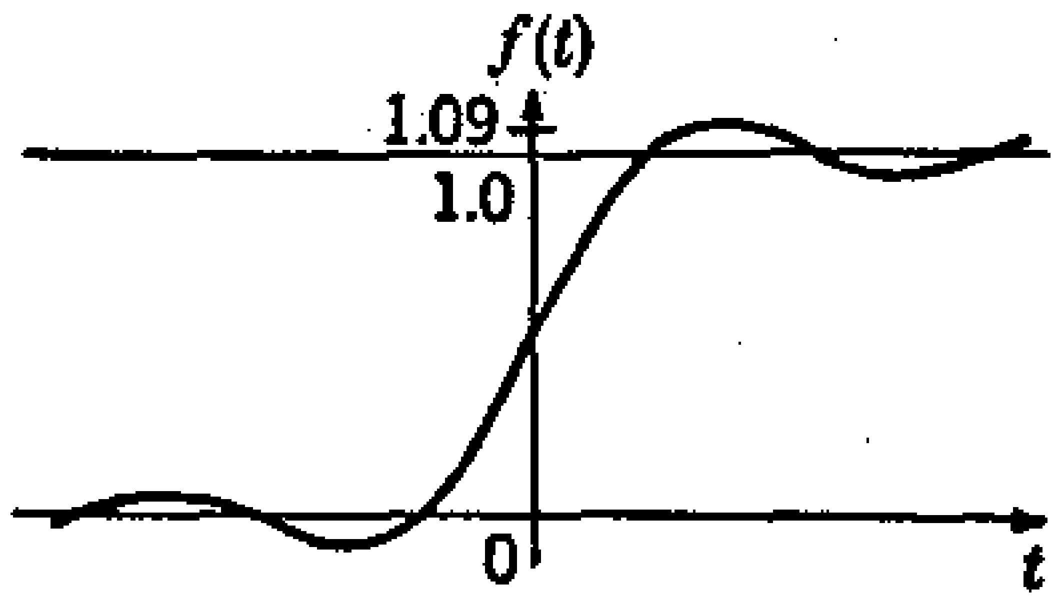


FIG. A·4.—Step-function response of idealized low-pass filter.

The reason for this difficulty lies in the false assumption that it is possible for a filter to have a gaussian-error, or idealized low-pass amplitude characteristic, and zero phase lag.

Moreover, the difficulty cannot be removed by associating a linear phase lag, no matter how great, with the gaussian-error or idealized low-pass amplitude characteristics. Although a linear phase lag has the effect of shifting the time origin in Figs. A·2 and A·4 to the left, no matter how great this shift to the left, it would still remain true that there would be some response even before the input switch were closed.

As a practical matter, a response of, for example, one-hundred-millionth per cent for time < 0 can be neglected, in the sense that it has no experimental meaning; hence these filters are "practically" realizable, with large enough phase lags (see Sec. A·3 for a more precise discussion of this point).

Nevertheless, it is of considerable mathematical interest to determine whether or not there is *any* phase function *whatever*, linear or otherwise, that can be associated with either the exact gaussian-error amplitude curve or the exact idealized low-pass amplitude curve so as to give *strictly zero* transient response for $t < 0$. The answer is no, and this negative answer is a consequence of an extraordinarily elegant theorem of Paley and Wiener,[1] which states, in the language of engineering:

If $A(\omega)$ is an arbitrary amplitude characteristic, i.e. an even non-negative function of frequency, having a Fourier transform, $A(\omega)$ is said to be "realizable" if it is possible to associate with the amplitude function $A(\omega)$ a phase-lag function $\phi(\omega)$ (not necessarily linear) such that the combined frequency function $A(\omega)e^{-j\phi(\omega)}$ yields zero transient response for $t < 0$. (This is clearly a very general and nonrestricted conception of realizability.) Then

Theorem of Paley and Wiener.—A necessary and sufficient condition for an amplitude function $A(\omega)$ to be realizable is that

$$\int_{-\infty}^{\infty} \frac{|\log A(\omega)|}{1 + \omega^2}\, d\omega \qquad (1)$$

be finite.[1]

In words: A realizable amplitude characteristic cannot have too great a total attenuation.

A realizable characteristic may have infinite rejection for a discrete set of frequencies, but it cannot have infinite rejection over a *band* of frequencies.

The theorem of Paley and Wiener has the great merit that although it is of a complex-variable nature in content and proof, Criterion (1) itself is entirely expressed in the domain of real variables.

Both the gaussian-error curve and the idealized bandpass characteristic attenuate too much to satisfy Criterion (1). Hence neither is exactly realizable. *This is not just a practical difficulty but a theoretical impossibility.*

The mathematical meaning of this nonrealizability is as follows: For every sequence of filters whose amplitude curves approximate more and more closely to the gaussian-error curve or to the idealized low-pass amplitude curve, it will be found that the successive phase characteristics diverge.

A·2. Examples. *Gaussian-error Curve.*—To illustrate this divergence of phase characteristics one may examine a familiar method of approxi-

The exact quotation is as follows: "Let $\phi(x)$ be a real nonnegative function not equivalent to zero, defined for $-\infty < x < \infty$, and of integrable square in this range. A necessary and sufficient condition that there should exist a real- or complex-valued function $F(x)$ defined in the same range, vanishing for $x \geqq x_0$ for some number x_0, and such that the Fourier transform $G(x)$ of $F(x)$ should satisfy $|G(x)| = \phi(x)$ is that

$$\int_{-\infty}^{\infty} \frac{|\log \phi(x)|}{1 + x^2}\, dx < \infty."$$

[1] Integral (1), if it exists, is in fact an evaluation for a special case of the expression giving the logarithm of a complex impedance function in terms of the logarithm of its absolute value.

mating in amplitude to a gaussian-error curve to see how it leads to divergent phase functions.

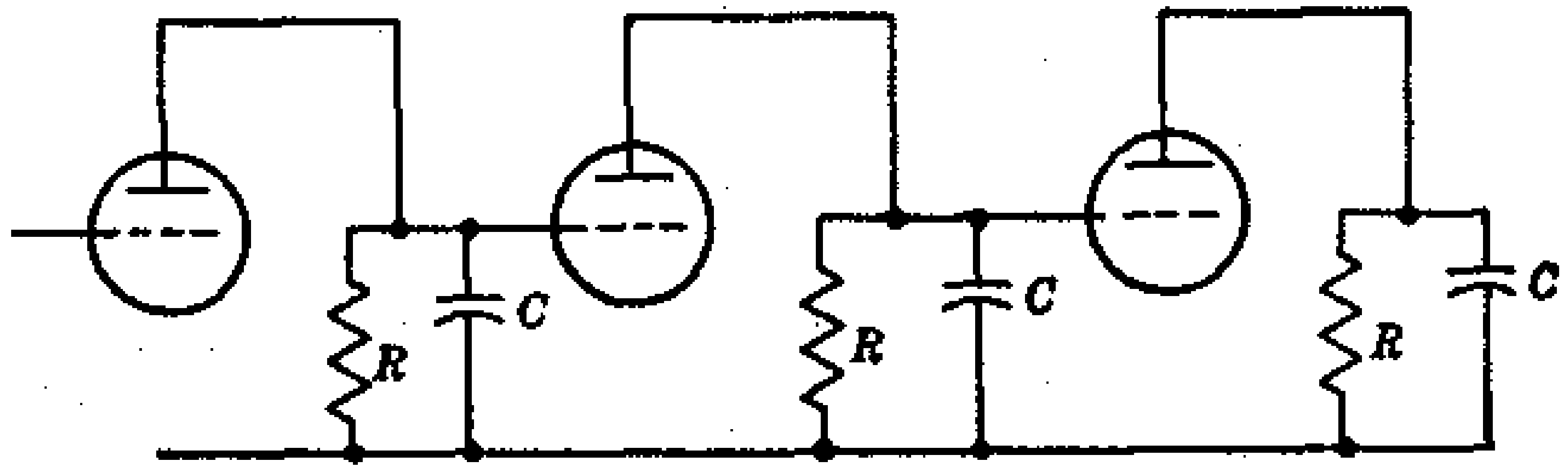


FIG. A·5.—Cascade of n identical RC-coupled pulse-amplifier stages.

Consider a cascade of n identical RC-coupled pulse-amplifier stages (Fig. A·5). The amplifier has the complex selectivity function

$$(1 + j\omega RC)^{-n}, \tag{2}$$

whose absolute value is $[1 + (\omega RC)^2]^{-\frac{n}{2}}$. If the over-all 3-db bandwidth is held constant as n increases, e.g., if the 0.707 voltage point is maintained at $\omega = 1$, then $RC = (2^{1/n} - 1)^{1/2}$. From the Taylor's series approximation $2^{1/n} - 1 \approx (\ln 2)/n$, Expression (2) becomes

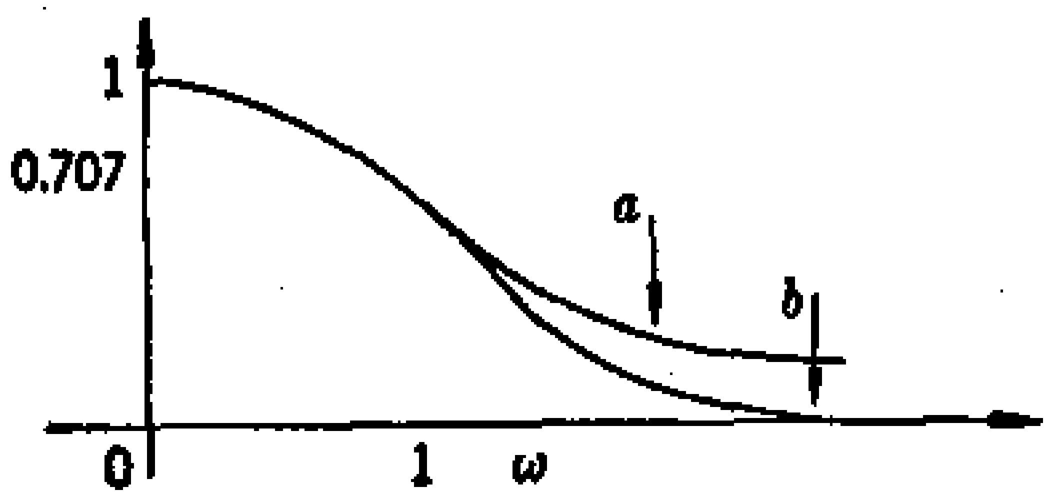


FIG. A·6.—Absolute value vs. frequency curves of (a) cascade of n identical RC-coupled stages, (b) gaussian-error curve; both curves adjusted to have their 3-db points at $\omega = 1$.

$$\left(1 + \frac{j\omega\sqrt{\ln 2}}{\sqrt{n}}\right)^{-n}. \tag{3}$$

The absolute value of Expression (3) is

$$A_n(\omega) = \left(1 + \frac{\omega^2 \ln 2}{n}\right)^{-\frac{n}{2}}, \tag{4}$$

and the phase lag of Expression (3) is

$$\phi_n(\omega) = n \tan^{-1} \frac{\omega\sqrt{\ln 2}}{\sqrt{n}}. \tag{5}$$

Compare Expression (4) with a gaussian-error curve having $\omega = 1$ as its 0.707 point (Fig. A·6), the formula for which is

$$e^{-\omega^2 \frac{\ln 2}{2}}.$$

It is plain from Expression (4) that $A_n(\omega)$ converges to $e^{-\omega^2\frac{\ln 2}{2}}$, as follows from the definition of e as $\lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$; on the other hand $\phi_n(\omega)$ diverges, by becoming indefinitely large for all ω. That is, the sequence of cascaded RC-coupled stages tends to a gaussian-error curve in amplitude, *but* (and this is the whole point of the example) the associated phase functions (5) diverge.

Idealized Low-pass Filter.—A similar state of affairs prevails for the idealized low-pass filter. This filter is not realizable (not because of the *steepness* with which the amplitude curve cuts off but rather because the amplitude curve cuts off to *zero*). The approximate low-pass filter

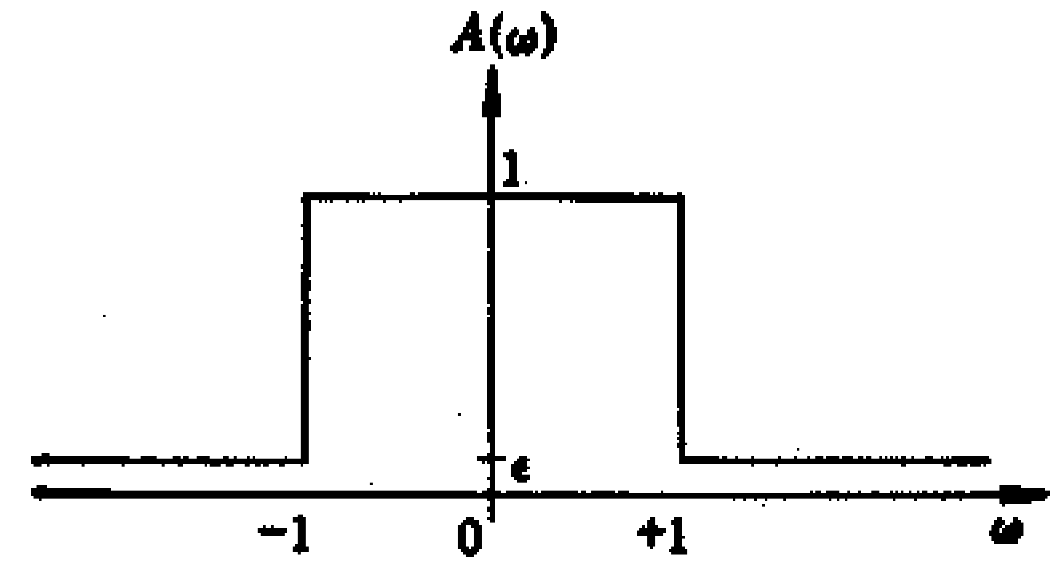


FIG. A·7.—Approximate low-pass filter, $A(\omega) = 1$ for $-1 < \omega < 1$, $= \epsilon$ otherwise.

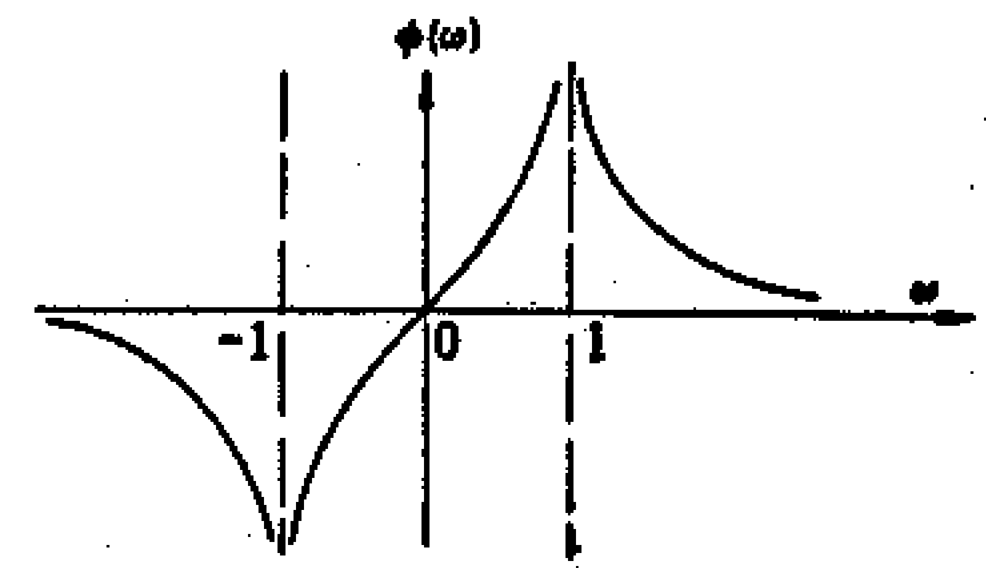


FIG. A·8.—Phase vs. frequency curve, $\phi(\omega) = |\ln \epsilon| \cdot \ln \left|\frac{1+\omega}{1-\omega}\right|$, corresponding to absolute-value curve of Fig. A·7.

of Fig. A·7, which cuts off to ϵ, is realizable, however, no matter how small ϵ may be. The corresponding phase-lag function is[1] (Fig. A·8) proportional to

$$\phi(\omega) = |\ln \epsilon| \ln \left|\frac{1+\omega}{1-\omega}\right|.$$

Filters Having $|\sin \omega/\omega|$ *and* $\sin^2 \omega/\omega^2$ *as Absolute Values.*—These two amplitude characteristics (Fig. A·9) do satisfy the Paley-Wiener criterion

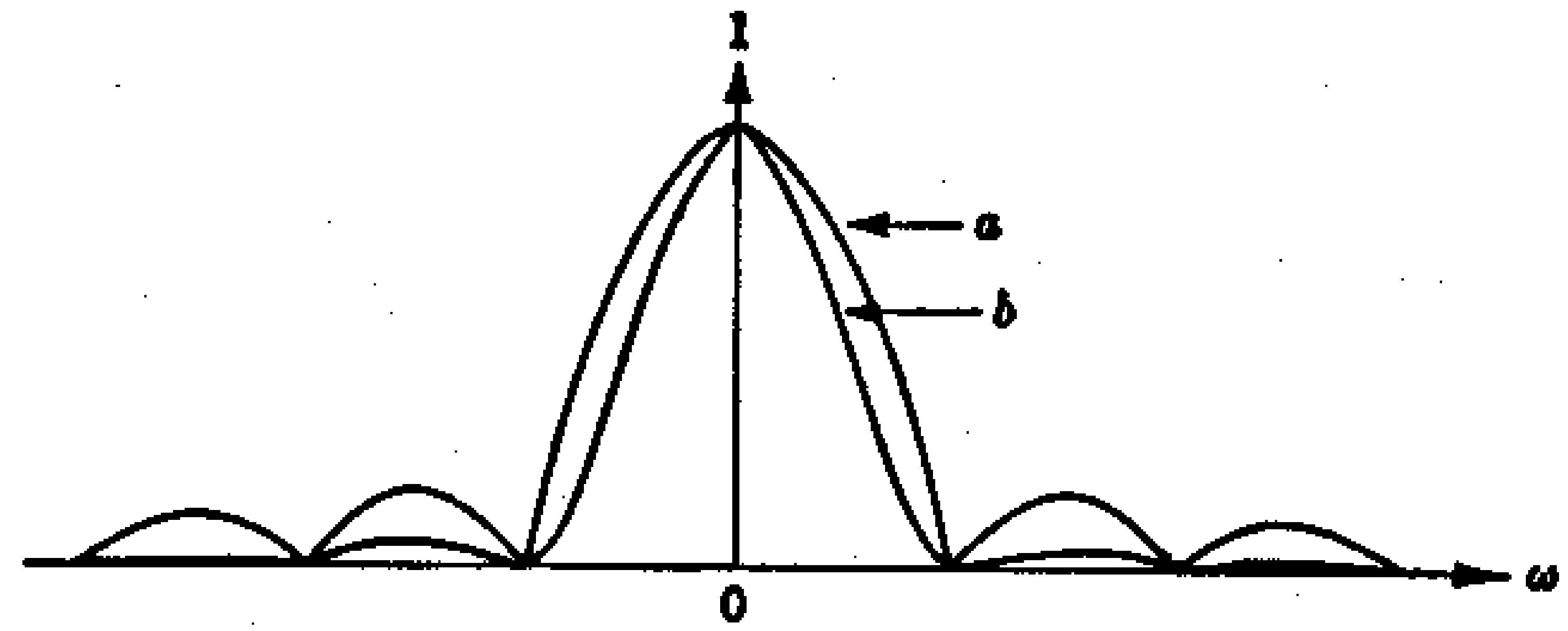


FIG. A·9.—Absolute value vs. frequency curve of (a) $(\sin \omega/\omega)$, and (b) $(\sin^2 \omega/\omega^2)$.

(1) and, as it happens, are very simply realizable, with linear phase lags, in the forms $(\sin \omega/\omega)e^{-j\omega}$ and $(\sin^2 \omega/\omega^2)e^{-2j\omega}$. Routine integration shows the step-function responses to be as shown in Fig. A·10. (The step-function response of the $(\sin^2 \omega/\omega^2)e^{-2j\omega}$ filter is made up of two parabolic arcs.)

The fact that $|\sin \omega/\omega|$ and $\sin^2 \omega/\omega^2$ satisfy Criterion (1) does not mean that these two amplitude curves, with their infinity of arches, are exactly realizable by means of a finite number of resistances, inductances, and capacitances, but it does mean that it is possible to have a sequence of filters, each made up of a finite number of lumped circuit elements, that tend to $|\sin \omega/\omega|$ or $\sin^2 \omega/\omega^2$ in amplitude and *whose phase functions converge* (as a matter of fact, converge to linear phase lags of ω and 2ω radians respectively).

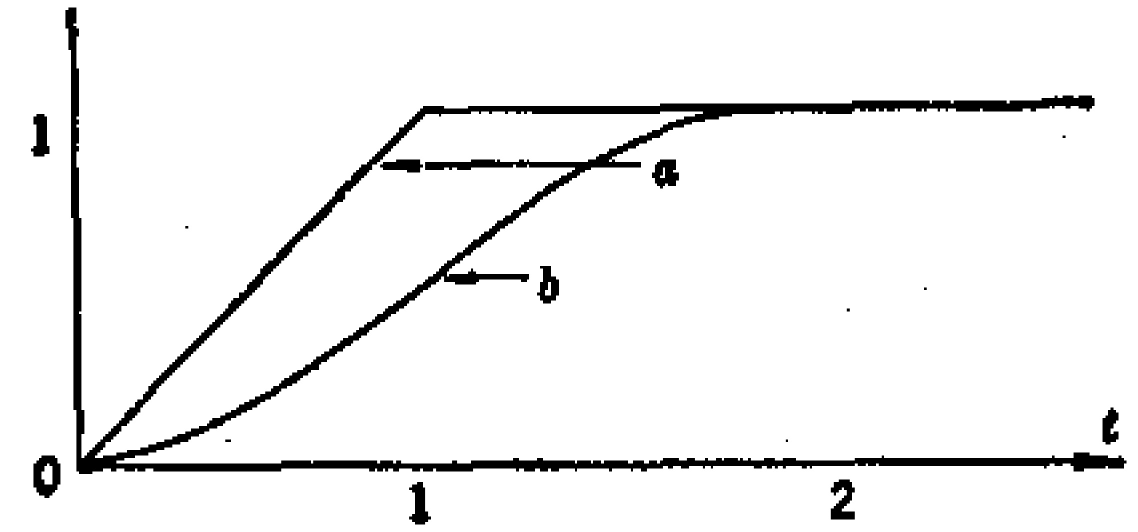


FIG. A·10.—Step-function response of (a) $(\sin \omega/\omega)e^{-j\omega}$ filter, and (b) $(\sin^2 \omega/\omega^2)e^{-2j\omega}$ filter.

Indeed, an example of such a sequence of finite filters has been suggested by E. A. Guillemin. In pulse-forming networks of the sort shown in Fig. A·11, as the number of elements increases, the current I_2 tends

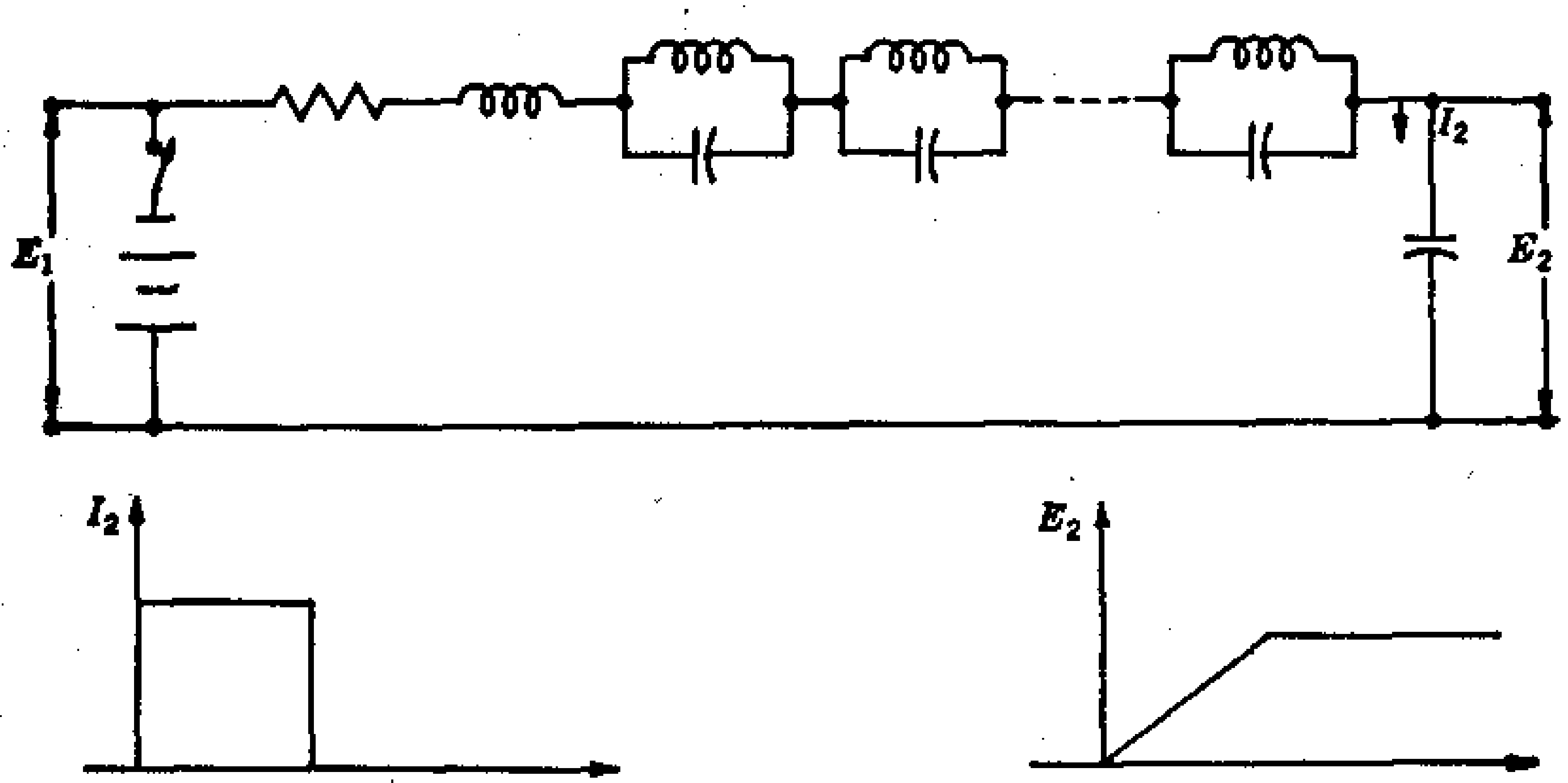


FIG. A·11.—Idealized pulse-forming network. The system function for E_2/E_1 is $(\sin \omega/\omega)e^{-j\omega}$.

more and more to a square pulse, and hence the voltage E_2 to an inclined step. Therefore the ratio E_2/E_1 (see Fig. A·10) tends to $(\sin \omega/\omega)e^{-j\omega}$.

It is even possible that the complex functions $(\sin \omega/\omega)e^{-j\omega}$ and $(\sin^2 \omega/\omega^2)e^{-2j\omega}$ can be exactly realized by a finite number of lines with distributed constants.

A·3. The Practical Meaning of the Paley-Wiener Criterion.—For sufficiently negative values of t, the Fourier transform of an arbitrary amplitude characteristic $A(\omega)$ [subject only to the (physically inevitable) condition of having a Fourier transform] is as small as is desired, whether or not the Paley-Wiener Criterion (1) is satisfied. Consequently it is possible to approximate to the given amplitude characteristic, as closely as desired over any finite frequency interval,[1] by a real filter with large phase lag, i.e., large time delay. If it happens that Criterion (1) is satisfied, then the theorem of Paley and Wiener asserts that there is, in fact, a way of carrying out this approximation so that the phase functions converge to some finite $\phi(\omega)$, thereby yielding a *finite* time delay.

The practical significance of the Paley-Wiener criterion is then as follows:

An amplitude characteristic having a Fourier transform can be approximated arbitrarily closely by finite filters whether it satisfies the Paley-Wiener criterion or not; but if it does satisfy the Paley-Wiener criterion, then the entire approximation process can be carried out within the bounds of a finite time delay, whereas if the Paley-Wiener criterion is not satisfied, the approximation process necessitates an infinite time delay.

The following may be said in summary: Amplitude characteristics having a Fourier transform can be divided into three types:

1. Those exactly realizable by a finite number of R, L, C elements, for example, $1/\sqrt{1 + \omega^2}$. Amplitude characteristics of this type satisfy the Paley-Wiener criterion, and in addition the corresponding complex frequency characteristic [$1/(1 + j\omega)$ for the above example] is rational.
2. Those exactly realizable by an infinite number of R, L, C elements (or by a finite number of lines with distributed constants), e.g., $|\sin \omega/\omega|$. These are exactly realizable in the sense that as the approximation to the given amplitude characteristic becomes closer and closer, the associated phase functions converge also. Amplitude characteristics of this type satisfy the Paley-Wiener criterion.
3. Those not exactly realizable at all but *approximable* arbitrarily closely over any finite frequency interval, although only at the expense of increasingly large time delay, e.g., $e^{-\omega^2}$. Amplitude characteristics of this type do not satisfy the Paley-Wiener criterion.

[1] But the approximating filter would then behave differently at $\omega = \infty$ and would therefore have an entirely different abstract-mathematical character.

APPENDIX B

CALCULATION OF LOAD-TUNING CONDENSER

A more refined treatment of the method of selecting the tuning condenser for the output stage of the two-stage amplifier (Sec. 9·6) can be considered if extremely small phase shift is desired. For a circuit with cathode- and plate-circuit impedances, the equivalent circuit can be

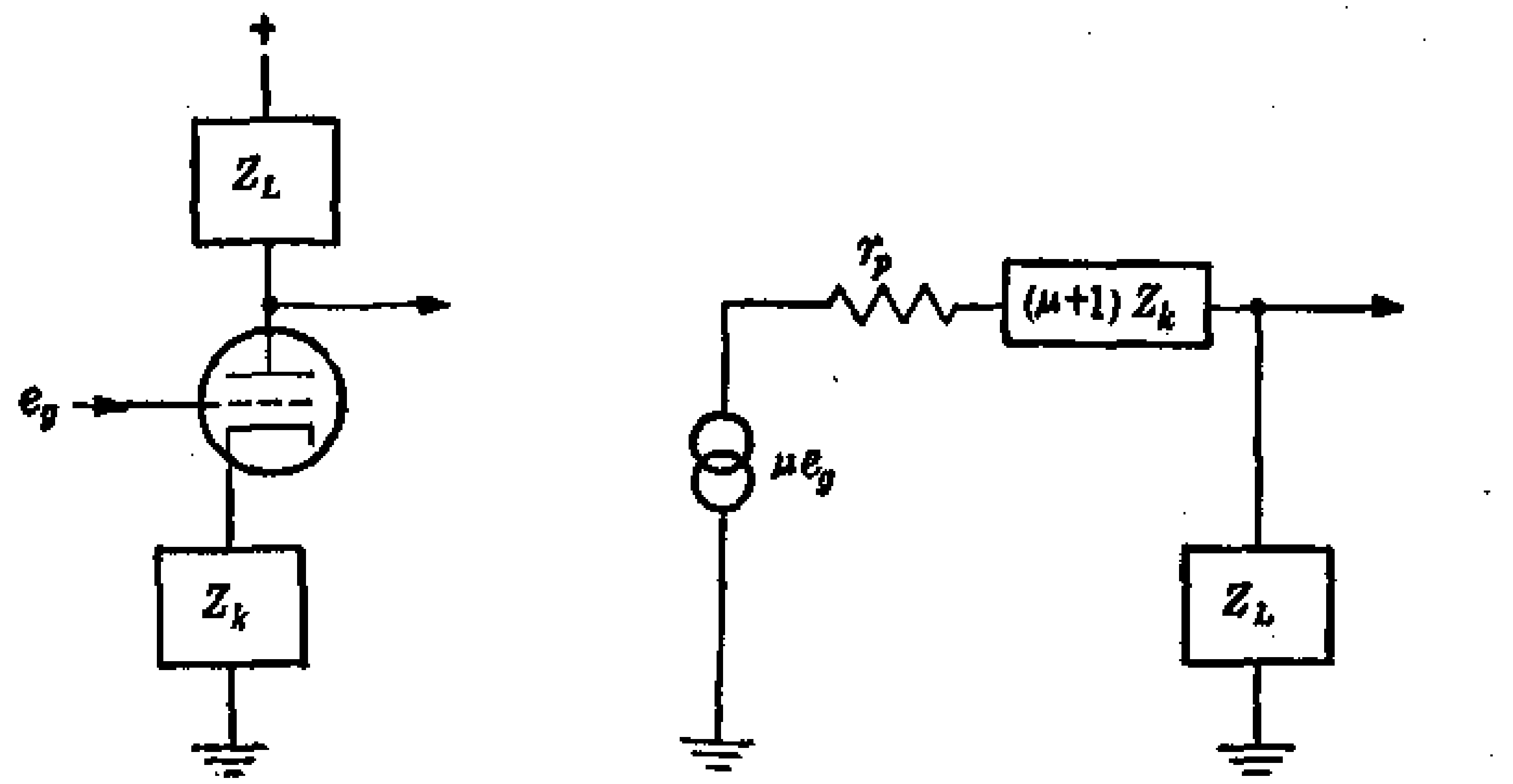


FIG. B·1.—Equivalent circuit of amplifier with impedance in cathode circuit.

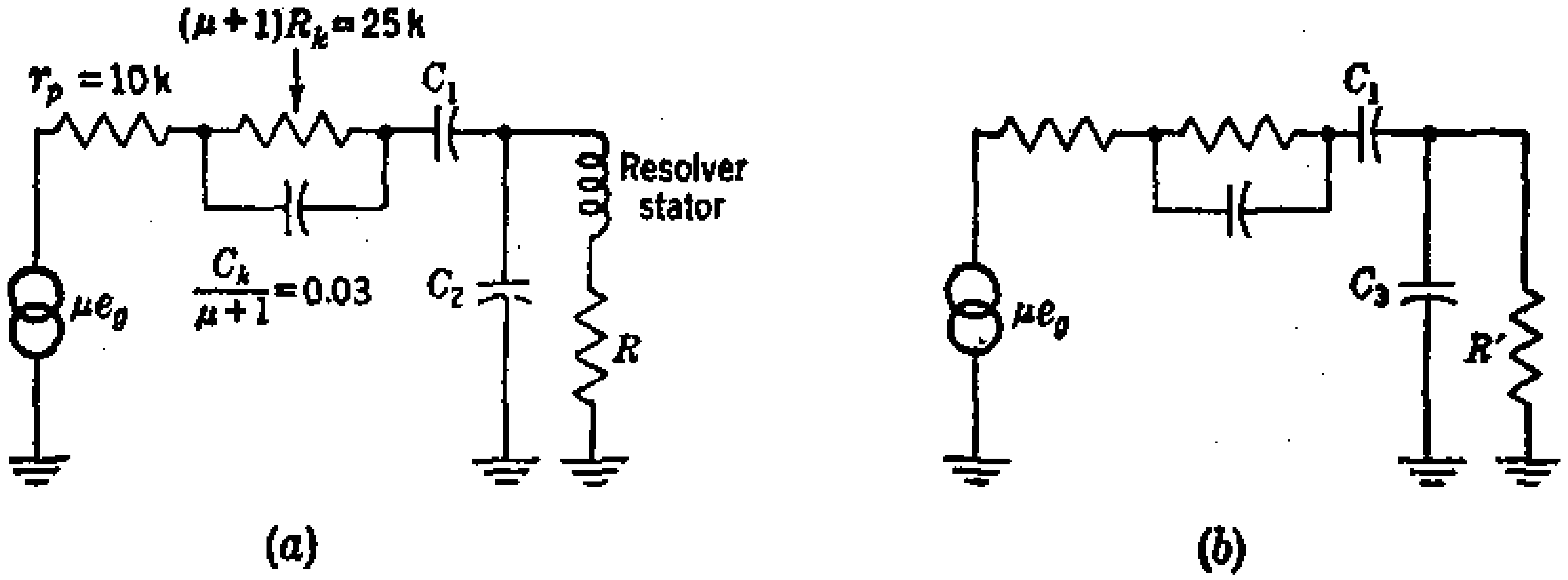


FIG. B·2.—Equivalent circuits of output stage. R' = apparent resistance of parallel-resonant circuit. C_3 = excess of C_2 over value required for resonant tuning. (a) Equivalent circuit of output stage; (b) simplified equivalent circuit if stator is tuned capacitatively.

drawn as is shown in Fig. B·1. For the operating point chosen, the 6C4 has $\mu \approx 16$, $r_p \approx 10{,}000$ ohms.

Therefore with the constants of Fig. 9·15*b* in the output stage, the equivalent circuit is as shown in Fig. B·2. The stage can be adjusted to zero phase shift if the load is tuned slightly capacitively. The magnitude of apparent capacitance can be determined as follows: The *LRC*-circuit may be considered as composed of a pure resistive tuned circuit (20*k*) in parallel with an additional capacitor C_3; if the cathode impedance is neglected (or if it is replaced by the corresponding series *RC*-circuit), the equivalent circuit of Fig. B·1 becomes part of a Wien bridge, and the condition for zero phase shift (in the case $Z_k \approx 0$) is $\omega^2 r_p R' C_1 C_3 = 1$; then,

$$C_3 = \frac{1}{(2\pi 500)^2 \times 17{,}000 \times 7500 \times 10^{-7}} = 0.008\ \mu\text{f}.$$

This circuit requires an extremely accurate selection of condensers, which may be feasible for experimental equipment but is impractical when wide production tolerances and temperature coefficients are involved.

APPENDIX C

DRIFT OF VACUUM-TUBE CHARACTERISTICS UNDER CONSTANT APPLIED POTENTIALS

The severest limitation is imposed upon direct-coupled amplifiers by the fact that the characteristics of any vacuum tube will gradually shift while the tube is in operation even though the conditions of operation are all held constant. For example, if the plate and grid voltages and filament or heater voltage are held constant, the plate current will drift slightly even after temperature equilibrium has been reached. This drift (except in the case of "microphonics" or dimensional changes due to shock or vibration) is attributable to cathode change resulting in variation of the average initial electron velocity of emission. This premise is substantiated by the nature of the shift of tube characteristics; it is found to be of the same type as that caused by variation of cathode temperature (see Fig. 11·7). Generally, as a tube ages, the drift is in the direction of decreasing emission; but short-time drifts are erratic as to sense and rapidity.

As in the case of cathode temperature effect (Sec. 11·6), the most convenient way to express the drift is in terms of the variation of control-grid bias that is required to hold the plate current constant with a given value of plate voltage (and screen-grid voltage). In addition to being more directly descriptive of the effect upon amplifiers, this quantity is more independent of plate current and voltage than is the drift of current with constant grid bias.

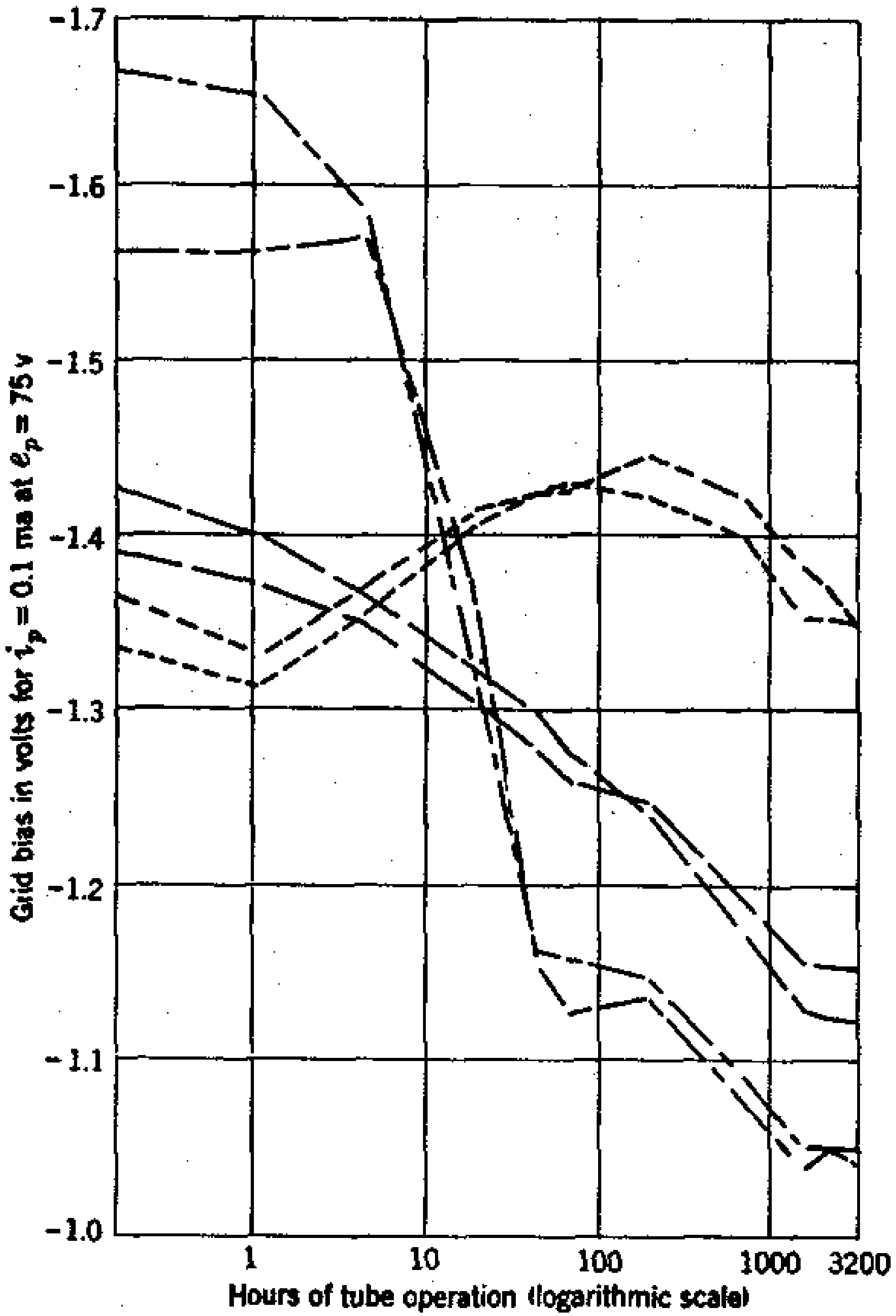
The first stage of an amplifier is usually the only stage wherein the tube drift is of much importance, since the drift of the second stage is less effective by a factor equal to the voltage gain of the first, etc. (Exceptions occur when the first stage is a cathode follower or an electrometer tube designed for low grid current but having very low gain.) This fact is fortunate, because the drift of a power stage is greater than that of a voltage amplifier operated at low power. Minimum drift in most ordinary receiving tubes occurs at plate currents of between 0.1 and 1 ma (10 to 100 μa for the small, filament types) and at plate and screen voltages as low as permissible from the standpoint of control-grid current, although neither current nor voltage is at all critical. The drift takes place as a general shift of the characteristics, as for cathode temperature change.

Rate of drift is relatively great in new tubes. Most of the drift during the life of a tube usually occurs during the first hundred hours of cathode operation. It is therefore advisable to age a tube, with filament voltage applied, for several days before use where stability is essential.

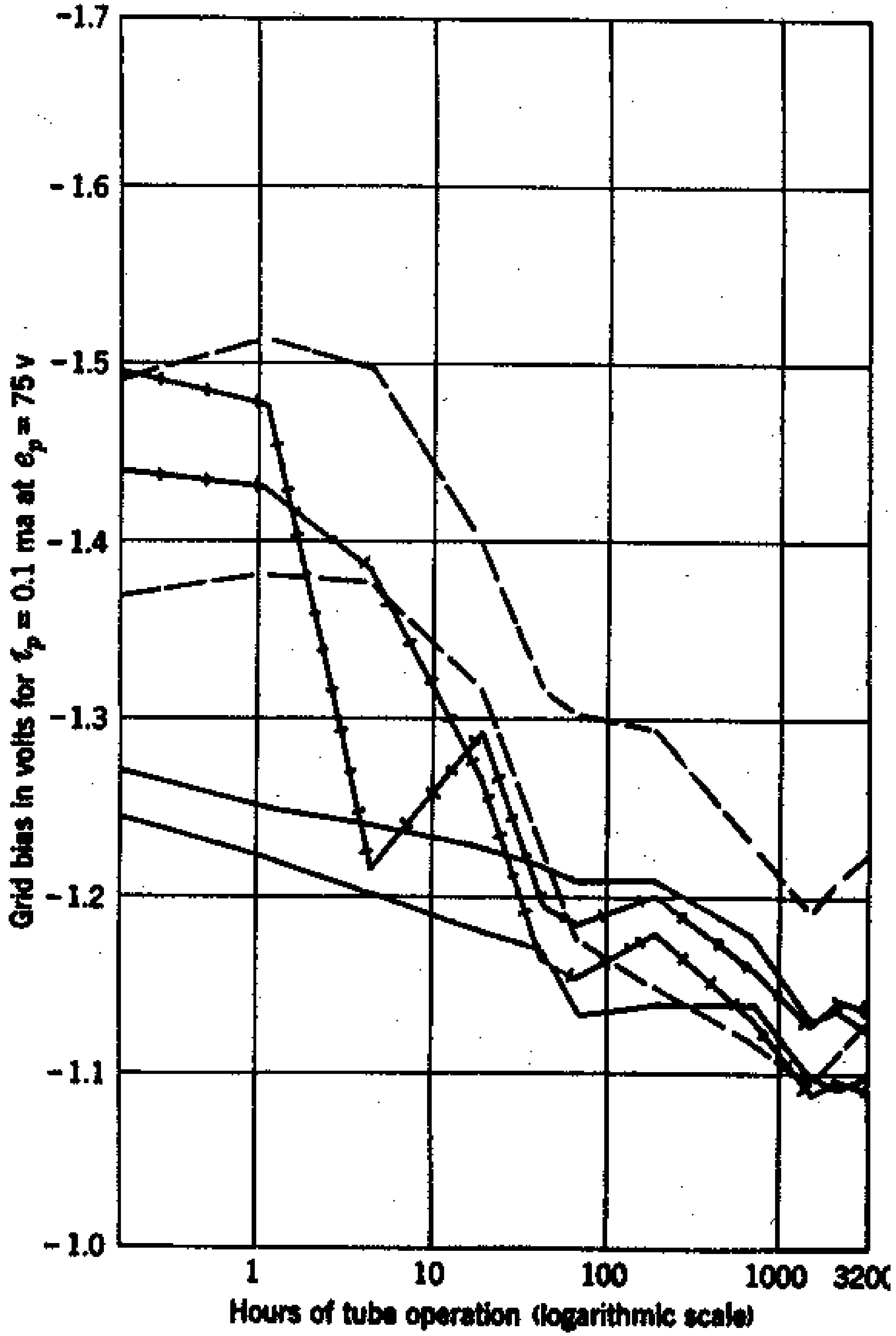
Figures C·1 and C·2 show the long-term drift of a group of 6SL7 double triodes. Readings of grid bias required, for a certain plate current at a certain voltage, were made at the times indicated. Heater voltage was held constant only during readings; the rest of the time it varied with the a-c line and was turned off several times. Because of the much more rapid variation at first, a logarithmic time scale is used.

Differential amplifiers are often used as input stages to help neutralize the effect of heater-voltage variation. The question arises as to whether any cancellation of drift is also accomplished. For short-time drift, where a zero adjustment can be made every day or so, the drift of each tube is erratic, and the use of two tubes instead of one in the first stage will multiply the probable drift by $\sqrt{2}$. For intervals of weeks or months, however, Figs. C·1 and C·2 seem to indicate that there is some drift cancellation between double triodes.

Figures C·1 and C·2 show that the amount of drift over periods of weeks is measured in tens of millivolts. (Some tubes are considerably worse than the group illustrated.) Thus, unless a zero adjustment can be made at frequent intervals, it is rather pointless to make an amplifier with a gain such that its output range is traversed as the input voltage changes by only 1 mv. If the zero set can be made just before the application, if a selected tube is used, and if heater voltage is constant, it is possible for input variations of a fraction of a millivolt to have significance.



DRIFT OF VACUUM-TUBE CHARACTERISTICS



FIGS. C·1 and C·2.—Drift of three 6SL7's (six triodes) during 3200 hr. Triode pairs are indicated by similar lineation.